On Nil Clean Group Rings

Y. Li[1]

[1]Department of Mathematics and Statistics
Brock University

July, 7, 2021

Brock University

Definition 1.1

An element of a ring is called nil clean if it is a sum of an idempotent and a nilpotent, and a ring is called nil clean if each element of R is nil clean.

Definition 1.1

An element of a ring is called nil clean if it is a sum of an idempotent and a nilpotent, and a ring is called nil clean if each element of R is nil clean.

Nil clean rings were introduced and related to clean rings by Diesl in 2006. Nil clean rings can be viewed as variants of well-known notion of clean rings, which are also closely related to boolean rings, strongly π-regular rings. The study of nil clean rings has recently attracted a great deal of attention, and one may refer to [1, 2, 6, 7, 10, 11, 12, 14, 15, 16] for more properties on nil clean rings.

In this talk , we will focus on the question of when a group ring RG is nil clean. We recall that group ring $RG = \{\sum_{g\in G} a_g g \mid a_g \neq 0 \text{ for finitely many } a_g\}$, where R is a commutative ring and G is a group with the operation $\cdot$. Then RG is a ring with respect to the addition defined by the rule

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

Introduction and Known Results

In this talk , we will focus on the question of when a group ring RG is nil clean. We recall that group ring $RG = \{\sum_{g\in G} a_g g \mid a_g \neq 0$ for finitely many $a_g\}$, where R is a commutative ring and G is a group with the operation $\cdot$. Then RG is a ring with respect to the addition defined by the rule

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g,$$

and the multiplication

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_g h) = \sum_{g\in G, h\in G} (a_g b_h) g \cdot h.$$

In 2015, McGovern, Raja and Sharp investigated commutative nil clean group rings.

Question 1.2

When a commutative group ring RG is nil clean?

In 2015, McGovern, Raja and Sharp investigated commutative nil clean group rings.

Question 1.2

When a commutative group ring RG is nil clean?

They completely settled this question.

Theorem 1.3 [12, Theorem 2.6]

A commutative group ring RG is nil clean if and only if R is nil clean and G is a 2-group.

In 2015, McGovern, Raja and Sharp investigated commutative nil clean group rings.

Question 1.2

When a commutative group ring RG is nil clean?

They completely settled this question.

Theorem 1.3 [12, Theorem 2.6]

A commutative group ring RG is nil clean if and only if R is nil clean and G is a 2-group.

It is natural to ask the following question:

It is natural to ask the following question:

Question 1.4

When a general group ring RG is nil clean?

In 2017, Sahinkaya, Tang and Zhou [14] investigated the nil cleanness of group rings in non-commutative cases.

In 2017, Sahinkaya, Tang and Zhou [14] investigated the nil cleanness of group rings in non-commutative cases.

- The group ring of a locally finite 2-group over a nil clean ring is nil clean.

In 2017, Sahinkaya, Tang and Zhou [14] investigated the nil cleanness of group rings in non-commutative cases.

- The group ring of a locally finite 2-group over a nil clean ring is nil clean.
- The group ring RS_3 is nil clean if and only if both R and the 2×2 matrix ring $M_2(R)$ are nil clean, where S_3 is the symmetric group of degree 3 (see [14]).

Other than the above mentioned results, very little is known about when a group ring of a non-abelian group is nil clean. In the talk, we investigate this case. In particular, we will focus on two types of groups: dihedral groups D_{2n} and generalized quaternion groups Q_{2n} [4].

The following lemmas will be used freely.

The following lemmas will be used freely.

Lemma 2.1 [7, Propositions 3.14 and 3.16]

If R is a nil clean ring, then $J(R)$ is nil and $2 \in J(R)$.

The following lemmas will be used freely.

Lemma 2.1 [7, Propositions 3.14 and 3.16]

If R is a nil clean ring, then $J(R)$ is nil and $2 \in J(R)$.

Lemma 2.2 [7, Proposition 3.15]

Let I be a nil ideal of a ring R. Then R is nil clean if and only if R/I is nil clean.

Nil Clean Group Rings

Lemma 2.3 [14, Theorem 2.6]

If RG is nil clean, then R is nil clean and $H(G)$ is a 2-group, where $H(G)$ is the hypercenter of G. Moreover, $R(G/H)$ is nil clean.

Remark 2.4

If RG is nil clean, then R is nil clean and $2R$ is nilpotent. Since char $R/2R = 2$, and $(R/2R)G$ is nil clean, we may assume char $R = 2$.

Nil Clean Dihedral Group Rings

We first investigate the nil cleanness of dihedral group rings RD_{2n} over a commutative ring R. First main result:

Theorem 2.5 [4, Theorem 2.3]

Let $n = 2^k m$ with $k \geqslant 0$, $(2, m) = 1$. The group ring RD_{2n} is nil clean if and only if, either $m = 1$ and R is nil clean, or $m = 3$ and RD_6 is nil clean.

Nil Clean Dihedral Group Rings

If $m = 1$, then $D_{2n} = D_{2^{k+1}}$ is a finite 2-group. By [14, Theorem 2.3], $RD_{2^{k+1}}$ is nil clean if and only if R is nil clean.

If $m = 1$, then $D_{2n} = D_{2^{k+1}}$ is a finite 2-group. By [14, Theorem 2.3], $RD_{2^{k+1}}$ is nil clean if and only if R is nil clean.
So we may assume that $m > 1$. The Forward Direction:

- Given that RD_{2n} is nil clean. By Remark 2.4, may assume Char $R = 2$.

Nil Clean Dihedral Group Rings

If $m = 1$, then $D_{2n} = D_{2^{k+1}}$ is a finite 2-group. By [14, Theorem 2.3], $RD_{2^{k+1}}$ is nil clean if and only if R is nil clean.
So we may assume that $m > 1$. The Forward Direction:

- Given that RD_{2n} is nil clean. By Remark 2.4, may assume Char $R = 2$.
- By Lemma 2.3, $R(D_{2n}/H)$ is nil clean, where $H = \langle g^{2^k} \rangle$ hypercenter of G.

Nil Clean Dihedral Group Rings

If $m = 1$, then $D_{2n} = D_{2^{k+1}}$ is a finite 2-group. By [14, Theorem 2.3], $RD_{2^{k+1}}$ is nil clean if and only if R is nil clean.
So we may assume that $m > 1$. The Forward Direction:

- Given that RD_{2n} is nil clean. By Remark 2.4, may assume Char $R = 2$.
- By Lemma 2.3, $R(D_{2n}/H)$ is nil clean, where $H = \langle g^{2^k} \rangle$ hypercenter of G.
- Note $R(D_{2n}/H) \cong RD_{2m}$, where $D_{2m} = \langle g_1, b \mid g_1^m = b^2 = 1, g_1^b = g_1^{-1} \rangle$.

- Consider $g_1 + g_1^{-1} \in RD_{2m}$. Since $g_1 + g_1^{-1}$ is nil clean, $(g_1 + g_1^{-1})^{2^l}$ is an idempotent.

Nil Clean Dihedral Group Rings

- Consider $g_1 + g_1^{-1} \in RD_{2m}$. Since $g_1 + g_1^{-1}$ is nil clean, $(g_1 + g_1^{-1})^{2^l}$ is an idempotent.
- $m = 3$ and RD_6 is nil clean.
- Using an induction argument and Lemma 2.2, we can easily prove the converse of the theorem.

Combining Theorem 2.5 with [14, Proposition 2.9], we have the following result.

Nil Clean Dihedral Group Rings

Combining Theorem 2.5 with [14, Proposition 2.9], we have the following result.

Corollary 2.6

The group ring RD_{2n} is nil clean if and only if either $m = 1$ and R is nil clean, or $m = 3$, and both R and $M_2(R)$ are nil clean.

One may note that if R is a commutative nil clean ring then $M_2(R)$ is nil clean (see [1, Corollary 7]). So the following result follows immediately.

Nil Clean Dihedral Group Rings

One may note that if R is a commutative nil clean ring then $M_2(R)$ is nil clean (see [1, Corollary 7]). So the following result follows immediately.

Corollary 2.7

If R is commutative, then RD_{2n} is nil clean if and only if $m = 1$ or 3, and R is nil clean.

Remark 2.8

We remark that in a recent published paper (2020) [8], the authors proved a result similar to Theorem 2.5 which characterizes when $\mathbb{Z}_2 D_{2n}$ is nil clean. However, our result is broader and our approach is different from theirs.

We next consider the case when $G = Q_{2n} = Q_{2^k m}$, with n even.

We next consider the case when $G = Q_{2n} = Q_{2^k m}$, with n even.

Theorem 2.9

The group ring RQ_{2n} is nil clean if and only if either $m = 1$ and R is nil clean, or $m = 3$ and RD_6 is nil clean.

Lastly, we provide an example of nil clean group ring with non nil-clean subring.

Nil Clean Generalized Quaternion Group Rings

Lastly, we provide an example of nil clean group ring with non nil-clean subring.

Example

The group ring RS_4 is nil clean if and only if RS_3 is nil clean. The subring RA_4 of RS_4 is never nil clean (as $\mathbb{Z}_2 A_4$ is not nil clean).

Nil $*$-clean Group Rings

In this section, we will focus on the nil $*$ cleanness of group ring RG.

Nil $*$-clean Group Rings

In this section, we will focus on the nil $*$ cleanness of group ring RG.

Definition 3.1

A ring R is called a $*$-ring (or a ring with involution $*$) if there exists an operation $*\colon R \to R$ such that

$$(x+y)^* = x^* + y^*, \ (xy)^* = y^* x^* \ \text{ and } (x^*)^* = x,$$

for all $x, y \in R$.

In this section, we will focus on the nil $*$ cleanness of group ring RG.

Definition 3.1

A ring R is called a $*$-ring (or a ring with involution $*$) if there exists an operation $*$: $R \to R$ such that

$$(x+y)^* = x^* + y^*, \ (xy)^* = y^* x^* \ \text{ and } (x^*)^* = x,$$

for all $x, y \in R$.

Example

The transpose is a $*$ involution of the $M_{n\times n}(\mathbb{R})$ ring.

Nil $*$-clean Group Rings

Definition 3.2

An element of a $*$-ring is called nil $*$- clean if it is a sum of a projection and a nilpotent, and the ring is called nil $*$-clean if each of its elements is nil $*$-clean.

Definition 3.2

An element of a $*$-ring is called nil $*$- clean if it is a sum of a projection and a nilpotent, and the ring is called nil $*$-clean if each of its elements is nil $*$-clean.

For a group rings RG, let the $* : RG \to RG$ be given by $(\sum a_g g)^* = \sum a_g g^{-1}$, which is an involution called the classical (or standard) involution on RG.

Commutative Nil $*$-clean Group Rings

The Main Result For Commutative Nil $*$-clean Group Rings.

The Main Result For Commutative Nil $*$-clean Group Rings.

Theorem 3.3

Let R be a commutative ring and G be an abelian group. The following are equivalent:
(1) RG is nil $*$-clean.
(2) RG is nil clean.
(3) R is nil clean and G is a 2-group.

Non-commutative Nil $*$-clean Group Rings

We next study the nil $*$-cleanness of the group ring RG over a non-abelian group G. As the nil cleanness of RD_{2n} as well as that of RQ_{2n} were characterized in the previous section, our focus will be on the case when $G = D_{2n}$ or $G = Q_{2n}$. The main result is different from the commutative setting.

Theorem 3.4

If R is a commutative ring, and $G = D_{2n}$ or Q_{2n} (with n even), then RG is nil $*$-clean if and only if R is nil clean and G is a 2-group.

Non-commutative Nil $*$-clean Group Rings

We next study the nil $*$-cleanness of the group ring RG over a non-abelian group G. As the nil cleanness of RD_{2n} as well as that of RQ_{2n} were characterized in the previous section, our focus will be on the case when $G = D_{2n}$ or $G = Q_{2n}$. The main result is different from the commutative setting.

Theorem 3.4

If R is a commutative ring, and $G = D_{2n}$ or Q_{2n} (with n even), then RG is nil $*$-clean if and only if R is nil clean and G is a 2-group.

Corollary 3.5

$\mathbb{Z}_2 S_3$ is nil clean, but not nil $*$-clean.

We close this section with the following example which provides a nil clean group ring (over a non-metacyclic group) which is not nil $*$-clean.

Example 3.6

The group ring $\mathbb{Z}_2 S_4$ is nil clean, but not nil $*$-clean.

Future Research Problems

Since D_{2n} and Q_{2n} are special types of metacyclic groups, we propose the following research problem for a broader class of group rings RG where G is a general metacyclic group.

Future Research Problems

Since D_{2n} and Q_{2n} are special types of metacyclic groups, we propose the following research problem for a broader class of group rings RG where G is a general metacyclic group.

Problem 1

Let $G = \langle a, b \mid a^n = b^m = 1, a^b = a^r \rangle$ be a metacyclic group. Characterize when the group ring RG is nil clean.

Future Research Problems

We note that for a finite abelian group G, if RG is nil clean, then G is a 2-group (i.e., 2 is the only prime divisor of $|G|$). However, this is not the case when G is non-abelian (e.g., $\mathbb{Z}_2 D_6$ is nil clean, but $3 \| D_6 | = 6$).

Future Research Problems

We note that for a finite abelian group G, if RG is nil clean, then G is a 2-group (i.e., 2 is the only prime divisor of $|G|$). However, this is not the case when G is non-abelian (e.g., $\mathbb{Z}_2 D_6$ is nil clean, but $3 \mid |D_6| = 6$).

Problem 2

If $p > 2$ is a prime, can we find a metacyclic group G such that $p \mid |G|$ and $\mathbb{Z}_2 G$ is nil clean?

We remark that the answer to Problem 2 is yes when $p = 3$ or 5. If the answer is no in general, we may ask the following question.

Future Questions

Research Problem 3

For what kind of prime $p > 5$, can we find a metacyclic group G such that $p || G|$ and $\mathbb{Z}_2 G$ is nil clean?

Thank You !

Bibliography I

- Breaz, S., Calugareanu, G., Danchev, P., Micu, T. (2013). Nil-clean matrix rings. *Linear Algebra Appl.* 439:3115-3119.
- Breaz, S., Modoi, G. C. (2016). Nil-clean companion matrices. *Linear Algebra Appl.* 489:50-60.
- Chen, H., Sheibani, A. M. (2019). Strongly 2-nil-clean rings with involutions. *Czechoslovak Math. J.* 69(144):317-330.
- Cui, J., Li, Y., Wang, H. (2021). On nil clean group rings, *Comm. Algebra* 49 (2021), no. 2, 790–796.
- Cui, J., Zhou, Y., Xia, G. (2019). Nil clean rings with involution, *Algebra Colloq.* in press.
- Diesl, A. J. (2006). *Classes of Strongly Clean Rings*. Ph.D. Thesis, Berkeley: University of California.

Brock University

Bibliography II

- Diesl, A. J. (2013). Nil clean rings, *J. Algebra* 383:197-211.
- Eo, S., Hwang, S., Yeo, W. (2020). Group rings satisfying nil clean property, *Commun. Korean Math. Soc.* 35(1):117-124.
- Gao, Y., Chen, J., Li, Y. (2015). Some $*$-clean group rings.*Algebra Colloq.* 22(1):169-180.
- Kosan, T., Wang, Z., Zhou, Y. (2016). Nil-clean and strongly nil-clean rings. *J. Pure Appl. Algebra* 220(2):633-646.
- Matczuk, J. (2017). Conjugate (nil) clean rings and Köthe problem. *J. Algebra Appl.* 16(4):1750073(14 pages).
- McGovern, W. W., Raja, S., Sharp, A. (2015). Commutative nil clean group rings. *J. Algebra Appl.* 14(6):1550094.

Bibliography III

Milies, C. P., Sehgal, S. K. (2002). *An introduction to Group Rings.* Kluwer Academic Publishers The Netherlands.

Sahinkaya, S., Tang, G., Zhou, Y. (2017). Nil-clean group rings. *J. Algebra Appl.* 16(7):1750135.

Ster, J. (2018). On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent. *Linear Algebra Appl.* 544:339-349.

Wu, Y., Tang, G., Zhou, Y. (2019). Nil-clean and unit-regular elements in certain subrings of $M_2(\mathbb{Z})$. *Czechoslovak Math. J.* 69(144):197-205.